→ Course website

Last time:

- Eigenvalues/vectors
- Proved the spectral theorem

HW due Saturday 7/17 at 11:59pm.

The spectral theorem gives a useful perspective on Fourier transforms. This is related to Pontryagin duality.

Thm: An operator on $\mathbb{C}^n$ is normal if and only if it is unitarily diagonalizable.

Here is a problem motivating the Discrete Fourier Transform.

Slogan: The properties of the Fourier transform are an application of the spectral theorem applied to the set of unitary operators inducing certain "permutations" of the basis vectors of our space.

Specifically, if our vector space is a space of functions $\psi: G \to \mathbb{C}$ for G an abelian group, then we consider the operators $\{T_g\}_{g \in G}$ which act via

$$(T_g \psi)(h) = \psi(g+h).$$

The spectral theorem gives us a new "basis", the conjugate basis or momentum basis or frequency basis, which unitarily diagonalizes all the (pairwise commuting) $\{T_g\}_{g \in G}$ operators.

This "basis" is indexed by (continuous) group homomorphisms $G \to U(1)$.

Examples: Fourier transform $G = \mathbb{R}$

Fourier series $G = S^1 = \mathbb{R}/2\pi\mathbb{Z}$

Discrete/Quantum Fourier Transform $G = \mathbb{Z}/N$

Let's consider the case $G = \mathbb{Z}/N$.

Then $V = \{f : G \to \mathbb{C}\}$

We have a basis

$$\{ |x\rangle \mid x \in G \}$$

where the function $|x_0\rangle$ is the function $G \to \mathbb{C}$ which is 0 when evaluated at $x \neq x_0$ and which is 1 when evaluated at $x = x_0$.

Define an inner product so that $\{|x\rangle\}_{x \in G}$ is an orthonormal basis.

Then we get the identity

$$\langle x | F \rangle = F(x)$$

for all $x \in G$.

We'll often write $|\psi\rangle$ to denote a function $G \to \mathbb{C}$ when doing quantum.

Our translation operators $\{T_g\}_{g \in G}$ are all powers of T_1 since G is cyclic.

We have $\langle x | T_1 | \psi \rangle = \langle x+1 | \psi \rangle$

In particular $\langle x | T_1 | x_0 \rangle = \langle x+1 | x_0 \rangle$

$= 0$ if $x \neq x_0 - 1$

1 if $x = x_0 - 1$

so T_1 is the operator T from the problem.

which is exactly the function $|x_0 - 1\rangle$.

Since T_1 is unitary, the spectral theorem implies there is an orthonormal basis for V consisting of eigenvectors of T_x. By the problem, there is (up to scalars) one eigenvector for each eigenvalue.

Let $\{ |\lambda\rangle \}_{\lambda \text{ is an eigenvalue of } T_1}$ be that orthonormal basis.

Then $T_1 = \sum_{\lambda} \lambda |\lambda\rangle\langle\lambda|$.

We can find $\langle x | \lambda \rangle$

$$= \langle \underbrace{T_1^{x} |x\rangle}_{=|0\rangle}, \underbrace{T_1^{x} |\lambda\rangle}_{= \lambda^{x} |\lambda\rangle} \rangle$$

$$= \lambda^{x} \langle 0 | \lambda \rangle \qquad \forall x.$$

Since $\{|x\rangle\}_{x\in G}$ is a basis of V, we find that $|\lambda\rangle$ is determined by the scalar $\langle 0|\lambda\rangle$.

We choose $\langle 0|\lambda\rangle = \frac{1}{\sqrt{N}}$ so that

$$|\lambda\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \lambda^x |x\rangle$$

is a normalized vector.

We have the identity

$$\langle x|\psi\rangle = \psi(x).$$

Now define

$$\tilde{\psi}(\lambda) \doteq \langle\lambda|\psi\rangle.$$

This a function which takes

an eigenvalue of T_1, equivalently
a group homomorphism $G \to U(1)$,
and outputs a complex number.

The function $\hat{\psi}: (\mathrm{Hom}(G, U(1))) \to \mathbb{C}$
is called the discrete Fourier transform
of $\psi: G \to \mathbb{C}$

In this case, $\mathrm{Hom}(G, U(1))$ can
be identified with $\{0, 1, \ldots, N-1\}$
since any $k \in \{0, \ldots, N-1\}$
gives an element

$$\exp\left(\frac{2\pi i}{N} k\right) \text{ in } U(1).$$

We identify $\phi: G \to U(1)$
with the unique $k \in \{0, \ldots, N-1\}$
such that $\phi(1) = \exp\left(\frac{2\pi i}{N} k\right)$.

We write $|k\rangle = |\lambda\rangle$, where $\lambda = \exp\left(\frac{2\pi i}{N} k\right)$ is an eigenvalue of T.

Thus $\langle x | k \rangle = \langle 0 | T^x | k \rangle = \lambda^x \langle 0 | k \rangle = \frac{1}{\sqrt{N}} \lambda^x$

$= \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} kx\right)$

Usually we think of the discrete Fourier transform

as taking $\psi : \mathbb{Z}/N = \{0, \dots, N-1\} \to \mathbb{C}$

to $\tilde{\psi} : \{0, \dots, N-1\} \longrightarrow \mathbb{C}$.

$\tilde{\psi}(k) = \langle k | \psi \rangle$

Here's how to compute the DFT:

Recall that $\sum_{x=0}^{N-1} |x\rangle\langle x| = I$

$$\tilde{\psi}(k) = \langle k | \psi \rangle = \langle k | I | \psi \rangle$$

$$= \langle k | \sum_{x=0}^{N-1} |x\rangle\langle x| \, |\psi\rangle$$

$$= \sum_{x=0}^{N-1} \langle k | x \rangle \langle x | \psi \rangle$$

$$= \sum_{x=0}^{N-1} \left(\langle x | k \rangle\right)^* \quad \psi(x)$$

complex conjugation

$$= \sum_{x=0}^{N-1} \left(\frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} kx \right) \right)^* \psi(x)$$

$$= \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \exp\left(-\frac{2\pi i}{N} kx \right) \psi(x)$$

Compare with

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \exp(-2\pi i k x) f(x) \, dx$$

for the usual Fourier transform on $G = \mathbb{R}$

For fourier transforms

$$(T_r f)(x) = f(x+r) \qquad x, r \in \mathbb{R}$$

$$\langle k | T_r f \rangle = \widetilde{(T_r f)}(k) = \exp(2\pi i k r) f(\omega)$$